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The exponential map for the conformal group $O(2, 4)$

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Abstract. We present a general method to obtain a closed finite formula for the exponential map from the Lie algebra to the Lie group for the defining representation of orthogonal groups. Our method is based on the Hamilton–Cayley theorem and some special properties of the generators of the orthogonal group and is also independent of the metric. We present an explicit formula for the exponential of generators of the $SO_+(p, q)$ groups with $p + q = 6$, in particular, dealing with the conformal group $SO_+(2, 4)$ which is homomorphic to the $SU(2, 2)$ group. This result is needed in the generalization of $U(1)$-gauge transformations to spin-gauge transformations where the exponential plays an essential role. We also present some new expressions for the coefficients of the secular equation of a matrix.

1. Introduction

The well known important formulae for the groups $SU(2)$ and $SO(3)$

$$\begin{aligned} e^{i(\theta/2)\sigma\cdot n} &= \cos(\theta/2)I_2 + i\sigma\cdot n\sin(\theta/2) \\ e^{\theta\mathcal{L}_j n_j} &= I_3 + \mathcal{L}_j n_j\sin\theta + (\mathcal{L}_j n_j)^2(1-\cos\theta) \end{aligned} \tag{1.1}$$

have been recently generalized to the group $SL(2, C)$ and its homomorphic group $SO_+(1, 3)$ (Zeni and Rodrigues 1990, 1992) but no such formulae seem to exist in the literature for the group $SO_+(2, 4)$. The purpose of the present work is to fill this gap, presenting an explicit finite formula for the series of the exponential of a g-skew-symmetric matrix (4.8) which represents the infinitesimal generators of the orthogonal group $SO_+(2, 4)$.

The group $SO_+(2, 4)$, or its covering $SU(2, 2)$, appears in several different contexts in theoretical physics. For instance, it is the invariance group of the bilinear invariants in the Dirac theory of the electron. It is also homomorphic to the relativistic conformal group, the largest group that leaves the Maxwell equations invariant (Bateman 1910, Cunningham 1909) or, in other words, it is the largest group which preserves the light cone of the Minkowski spacetime (Gürsey 1956, Barut 1971). For a good review see Fulton *et al* (1962). Other applications are found in the study of dynamical groups (Barut 1972).

More recently, the group $SU(2, 2)$, or its subgroups, has been used in spin-gauge theories in an attempt to generalize the minimal coupling and to unify electrodynamics and gravitation (Dehnen and Ghaboussi 1986, Chisholm and Farewell 1989, Liu 1992, Barut and McEwan 1984) and in conformally compactified spacetimes (Barut *et al* 1994).

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The mathematical problem can be stated basically as the sum of the exponential series for the matrix representing the generators of orthogonal groups. The Hamilton–Cayley theorem plays a key role in solving this problem since it gives a recurrence relation between the powers of the matrix and, hence, we can transform the matrix series into a real-number series.

The exponential map, as well as other types of parametrization for the group elements of unitary groups, in particular $SU(3)$ and $SU(4)$, deserves more attention in the literature since unitary groups play an important role in quantum mechanics and particle physics (see Barnes and Delbourgo (1972) and references therein) and, in particular, the work of Bincer (1990) on unitary groups, which presents a parametrization of the exponential map through a set of orthonormal vectors obtained by considering the diagonal form of the generators. The work of Bincer (1990) is related to the Jordan-form method of constructing the exponential since the latter provides a parametrization of the exponential by the eigenvectors of the matrix (Faria-Rosa and Shimabukubo 1993).

There are several articles in the literature concerned with the exponential of an arbitrary matrix, for instance the work of Moler and van Loan (1978) has a comprehensive review of methods, analytical and numerical, for dealing with the exponential of an arbitrary matrix as well as an extensive list of references.

We remark that besides the exponential map there are other possible parametrizations of group elements using the Lie algebra (see, for example, Lounesto (1987) for the Cayley map). However, the exponential map deserves special attention due to its relationship with systems of differential equations as discussed below.

The symmetry properties of the matrices representing the generators of orthogonal groups are also helpful as they allow us to separate the series into even and odd powers. It is a remarkable characteristic of orthogonal groups that either even or odd powers occur in the Hamilton–Cayley theorem which amounts to a great simplification for the sum of the exponential series. These facts are discussed in sections 2 and 3 where general recurrence relations are obtained for the powers of the generators of orthogonal groups.

An important step in summing the series of a matrix is to consider the eigenvalues of the matrix instead of the coefficients which appear in the Hamilton–Cayley theorem. We expect the series for the exponential of a matrix to be expressed by means of elementary functions of eigenvalues of the matrix because if we consider the solutions of a system of first-order differential equations

$$\frac{\mathrm{d}X}{\mathrm{d}t} = HX \tag{1.2}$$

the solutions are given by the exponential of H parametrized by t, i.e. $X(t) = \mathrm{e}^{Ht}X_0$ (see Magnus (1954) and Fer (1958) for the cases where H is a function of time). On the other hand, we can express the components of the vector $X(t)$ as exponential (scalar) functions of the eigenvalues as given below

$$X_j(t) = \mathrm{e}^{\lambda_k t} C_{jk} \tag{1.3}$$

where the C_{jk} are chosen to fit the initial value X_0.

Therefore, if we compare both solutions, it is obvious that the matrix elements of the exponential of a matrix must be elementary functions of the eigenvalues. Another way to see that the above assertion holds is to look at the Jordan form of the matrix H (Faria-Rosa and Shimabukubo 1993).

In section 4 we derive an explicit finite formula for the exponential of a matrix representing the generator of the $SO_+(p,q)$ group with $p+q=6$. In order to close the series easily we have used the discriminant related to the secular equation. The series for the exponential is written as a product of elementary functions of the eigenvalues by the first few powers of H. We remark that the series is obtained in an intrinsic way, without explicit reference to the matrix elements.

In section 5 we introduce a matrix representation for the generators of the $SO(2,4)$ and some formulae which simplify the previous results, especially a new expression for the odd powers. We specialize the previous result for the Lorentz group recovering the result of Zeni and Rodrigues (1990) and also equation (1.1).

In the appendix we discuss a new method for obtaining the coefficients of the secular equation from the trace of the powers of the matrix.

2. The generators of orthogonal groups

The matrices related to the defining representation of the orthogonal groups $O(p,q)$ are defined by the following condition:

$$A^t g A = g \qquad \text{or} \qquad g A^t g = A^{-1} \tag{2.1}$$

where g is a diagonal matrix with p entries equal to $+1$ and q entries equal to -1 and the superscript t indicates the transposed matrix.

The connected component of the identity of the orthogonal groups will be hereafter indicated by $SO_+(p,q)$. In what follows, we are concerned with those transformations $A \in SO_+(p,q)$ that can be written as e^H where H is called an infinitesimal generator of the group, i.e. an element of the Lie algebra (Miller 1972, Barut and Raczka 1986).

Equation (2.1) shows that the generators of the orthogonal group are given by

$$gH^t g = -H \tag{2.2}$$

since we have that $g e^{H^t} g = e^{gH^t g}$.

The generators of the orthogonal group will be called here g-skew-symmetric null-diagonal matrices. The number of independent real parameters is $(n^2-n)/2$ which corresponds to the number of elements in the upper (or lower) triangular matrix.

2.1. Matrix symmetry of the powers of the generators

Lemma. The odd powers of H are again g-skew-symmetric null-diagonal matrices. On the other hand, the even powers of H are g-symmetric matrices, i.e.

$$H^{2n} = g(H^t)^{2n} g \qquad \text{while} \qquad H^{2n+1} = -g(H^t)^{2n+1} g. \tag{2.3}$$

This result simplifies the task of finding a finite closed form for the exponential since it shows that we can work separately with the series of even and odd powers

$$A = e^H = \frac{1}{2}(A + gA^t g) + \frac{1}{2}(A - gA^t g) = \sum_{n=0}^{\infty} \frac{H^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{H^{2n+1}}{(2n+1)!}.$$

We will show that the recurrence relations for both series of even and odd powers are similar, i.e. we can deduce one from another.

2.2. Parity of the secular equation

We consider now the secular equation for g-skew-symmetric matrices

$$\det(H - \lambda I_n) = 0. \tag{2.4}$$

Lemma. Let H be an $n \times n$ g-skew-symmetric matrix. If n is odd (even) then only odd (even) powers of the eigenvalues are present in the secular equation (Turnbull (1960) presents a proof for the Euclidean case but the result holds for every metric), i.e.

$$\det(H - \lambda) = (-1)^n(\lambda^n + C_2\lambda^{n-2} + C_4\lambda^{n-4} + \cdots + C_{n-x}\lambda^x) \tag{2.5}$$

where $x = 0$ if n is even and $x = 1$ if n is odd.

The lemma follows from the identity below which can be proved by using the g-skew-symmetry of H (2.2):

$$\det(H - \lambda) = (-1)^n \det(H + \lambda). \tag{2.6}$$

Thus it is clear that the determinant has a defined parity under the change of λ to $-\lambda$.

3. The Hamilton–Cayley theorem

The Hamilton–Cayley theorem guarantees that a matrix satisfies a matrix equation as its secular equation, i.e. for the generators of the orthogonal groups we have

$$H^n + C_2H^{n-2} + C_4H^{n-4} + \cdots + C_{n-x}H^x = 0 \tag{3.1}$$

where the Cs are the coefficients of λ in the secular equation (2.5).

According to equation (3.1), we can express H^n in the case of even n in terms of the following set of matrices $H^{n-2}, H^{n-4}, \ldots, I_n$. By iterating equation (3.1) by H^2 we can express all even powers of H in terms of the same set of matrices as discussed below.

Also, we can apply an analogous reasoning to express all the odd powers of $H_{n\times n}$, with n even, by means of the set $H^{n-1}, H^{n-3}, \ldots, H$.

3.1. Recurrence relations for the powers of the generators

In the following, we consider $n \times n$ matrices with even n, i.e. generators of the groups $O(2m)$, $2m = n$, and consider the recurrence relations for even powers. In the case of the groups $O(2n+1)$, the recurrence relations shown below can be obtained in an analogous way.

Let us change our previous notation and write the recurrence relation resulting from secular equation (3.1) as follows.

$$H^n = a_0H^{n-2} + b_0H^{n-4} + \cdots + v_0H^2 + x_0I_n \tag{3.2}$$

where we are considering that n is even and I_n denotes the $n \times n$ identity matrix. In general, we set ($k \geqslant 0$)

$$H^{n+2k} = a_kH^{n-2} + b_kH^{n-4} + \cdots + v_kH^2 + x_kI_n. \tag{3.3}$$

We are going to determine a recurrence relation for the coefficients $a, b, \ldots, x$ present in the previous equation. To do this we iterate the equation for H^{n+2k}, multiplying it by H^2, and, after substituting equation (3.2) for H^n, we obtain

$$a_{k+1} = a_0 a_k + b_k \qquad b_{k+1} = a_k b_0 + c_k \qquad \ldots \qquad v_{k+1} = a_k v_0 + x_k \qquad x_{k+1} = a_k x_0 \tag{3.4}$$

The recurrence relations are the same for the coefficients of even and odd powers provided that we define for the odd powers the following formula (n even)

$$H^{n+1+2k} = a_k H^{n-1} + b_k H^{n-3} + \cdots + v_k H^3 + x_k H. \tag{3.5}$$

A closer analysis of equation (3.4) shows that if we determine the coefficient a_k then all other coefficients will be determined. The key to this problem is that the last coefficient x_k is expressed by means of a_k. Furthermore, we can get a recurrence relation for a_k only involving a_j $(j < k)$ by substituting the recurrence relations for b_k in the expression for a_k.

4. The orthogonal groups $O(p, q)$, $p + q = 6$

For the orthogonal groups $O(p,q)$ where $p+q=6$, the secular equation is a third-order polynomial in H^2

$$H^6 - a_0 H^4 - b_0 H^2 - c_0 I_6 = 0. \tag{4.1}$$

The previous recurrence relations for the coefficients a_k, b_k and c_k (3.4) can now be put in a simpler form which ensures that if we determine the coefficients a_k then the others are determined, as mentioned before

$$\begin{aligned} c_{k+1} &= a_k c_0 \qquad k \geqslant 0 \\ b_{k+1} &= a_k b_0 + a_{k-1} c_0 \qquad k \geqslant 1 \\ a_{k+1} &= a_k a_0 + a_{k-1} b_0 + a_{k-2} c_0 \qquad k \geqslant 2 \end{aligned} \tag{4.2}$$

where the first values follow from equation (3.4)

$$a_1 = a_0^2 + b_0 \qquad b_1 = a_0 b_0 + c_0 \qquad a_2 = a_1 a_0 + a_0 b_0 + c_0. \tag{4.3}$$

Our goal is to get a general formula for the coefficients a_k, b_k, c_k from these recurrence relations in terms of the eigenvalues. For this purpose, we express the coefficients a_0, b_0 and c_0, that appear in the secular equation (4.1), in terms of the eigenvalues which are the roots of that equation. We remark that the secular equation is a third-order equation in the square of the eigenvalues, so that the eigenvalues appear in pairs that we will indicate by $\{\pm x, \pm y, \pm z\}$. We are going to use only three eigenvalues $\{x, y, z\}$. Therefore, we have

$$\begin{aligned} a_0 &= x^2 + y^2 + z^2 \\ b_0 &= -x^2 y^2 - x^2 z^2 - y^2 z^2 \\ c_0 &= x^2 y^2 z^2. \end{aligned} \tag{4.4}$$

We shall also make use of the multiplier w to simplify the expression for the recurrence relations. w is equal to the square root of the discriminant of the secular equation and is given by

$$w = (x^2 - y^2)(z^2 - x^2)(y^2 - z^2). \tag{4.5}$$

Writing the above recurrence relation for a_k (4.2) by means of x, y and z, and after multiplying by w, we find the following final form of the recurrence relations

$$wa_k = (x^2 - z^2)y^{2k+6} + (z^2 - y^2)x^{2k+6} + (y^2 - x^2)z^{2k+6} \tag{4.6}$$

which can be proven by finite induction and holds for $k \geqslant 0$ in spite of the fact that equation (4.2) holds only for $k \geqslant 2$.

Substituting the above expression for wa_k into the recurrence relation for b_k (4.2), we get the following recurrence relation for the product wb_k:

$$wb_k = (z^4 - x^4)y^{2k+6} + (y^4 - z^4)x^{2k+6} + (x^4 - y^4)z^{2k+6} \tag{4.7}$$

which holds again for $k \geqslant 0$.

4.1. The series for the exponential $O(p, q)$, $p + q = 6$

We have seen that the series for the exponential of the generators of orthogonal groups can be conveniently divided into the series for even and odd powers. Moreover, each series can be expressed by means of only a few powers of the generator. Summarizing our previous result, we have transformed the matrix series into a real-number series for the coefficients a_k, b_k and c_k for which recurrence relations were obtained in the last section for the groups $O(p, q)$, $p + q = 6$.

After substituting equation (3.3) for H^{6+2k}, the series of even powers of H, multiplied by w, can be written as

$$I_6\left(w+\sum \frac{wc_k}{(6+2k)!}\right)+H^2\left(\frac{w}{2}+\sum \frac{wb_k}{(6+2k)!}\right)+H^4\left(\frac{w}{4}+\sum \frac{wa_k}{(6+2k)!}\right)$$

where all the sums run from zero to infinity.

We can also write an analogous expression for the series of odd powers.

Considering the previous recurrence relations (4.6), (4.7) and (4.2), the series of even and odd powers can be summed easily and the result for the exponential of H, multiplied by w, is given by

$$\begin{aligned} we^H = {} & [(y^4 - z^4)\cosh x + (z^4 - x^4)\cosh y + (x^4 - y^4)\cosh z]H^2 \\ & + [(z^2 - y^2)\cosh x + (x^2 - z^2)\cosh y + (y^2 - x^2)\cosh z]H^4 \\ & + [(x^2 - z^2)x^2z^2\cosh y + (z^2 - y^2)y^2z^2\cosh x + (y^2 - x^2)x^2y^2\cosh z]I_6 \\ & + \left[(x^2 - z^2)x^2z^2\frac{\sinh y}{y} + (z^2 - y^2)y^2z^2\frac{\sinh x}{x} + (y^2 - x^2)x^2y^2\frac{\sinh z}{z}\right]H \\ & + \left[(z^4 - x^4)\frac{\sinh y}{y} + (y^4 - z^4)\frac{\sinh x}{x} + (x^4 - y^4)\frac{\sinh z}{z}\right]H^3 \\ & + \left[(x^2 - z^2)\frac{\sinh y}{y} + (z^2 - y^2)\frac{\sinh x}{x} + (y^2 - x^2)\frac{\sinh z}{z}\right]H^5. \end{aligned} \tag{4.8}$$

The above series are valid for all groups $SO_+(p, q)$ with $p+q = 6$. The number of possible imaginary eigenvalues distinguishes the metrics and so transforms some of the hyperbolic functions given above into trigonometric functions. For example, all eigenvalues of the generator H are imaginary for the group $O(6, 0)$.

In the next section we are going to introduce some particular cases of the above formula.

5. The $SO_+(2,4)$ group

At this point in the discussion we want to focus on the group $SO_+(2,4)$. A generic generator of the group $SO_+(2,4)$ can be represented by the following matrix:

$$H = h_{ij}\mathcal{L}^{ij} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 & v_0 & a_0 \\ e_1 & 0 & b_3 & -b_2 & v_1 & a_1 \\ e_2 & -b_3 & 0 & b_1 & v_2 & a_2 \\ e_3 & b_2 & -b_1 & 0 & v_3 & a_3 \\ v_0 & -v_1 & -v_2 & -v_3 & 0 & c \\ -a_0 & a_1 & a_2 & a_3 & c & 0 \end{pmatrix} \tag{5.1}$$

where the metric is given by $g = \mathrm{diag}(+,-,-,-,-,+)$. h_{ij} $(i \leqslant j)$ are the matrix elements of H and $\mathcal{L}_{ij}$ is the standard basis for the Lie algebra of $SO_+(2,4)$ in the defining representation, i.e. they are matrices with only two non-zero g-skew-symmetric elements which are equal to ± 1 (Barut 1971).

The form above for the matrix generator, and therefore also for the metric, was chosen in such a way that makes it possible to establish a closer connection with the generators of $SU(2,2)$ which can be represented by Dirac matrices (Barut 1964, Kilhberg *et al* 1966). For example, the components v_μ and a_μ, $\mu \in [0,3]$ are related to a vector and axial vector in Dirac algebra. On the other hand, we set the generator of $SO_+(2,4)$ in such a way that one of its subgroups, the Lorentz group $SO_+(1,3)$, has a priviledged place; it corresponds to the 4×4 block formed by the e_j and b_j, $j = 1,2,3$. The last component of the generator above c corresponds to the generator of a chiral transformation in the Dirac algebra, i.e. a transformation generated by γ_5.

The coefficients C_k, present in the secular equation, can be written as (see also appendix, equations (A.4) and (A.5))

$$\begin{aligned} C_2 &= -\tfrac{1}{2}h_{ij}h_{ji} = -\tfrac{1}{2}\,\mathrm{Tr}\,H^2 \\ C_4 &= -\tfrac{1}{2}p_{ij}p_{ji} = -\tfrac{1}{2}\,\mathrm{Tr}\,P^2 \\ C_6 &= \det H = [\tfrac{1}{6}\,\mathrm{Tr}\,H\cdot P]^2 = [\tfrac{1}{6}h_{ij}p_{ji}]^2 \end{aligned} \tag{5.2}$$

where we sum over repeated indices. The elements p_{ij} of the matrix P are defined below.

We remark that the determinant of a skew-symmetric matrix is equal to the square of a polynomial which defines the Pfaffian of the matrix.

The elements p_{ij} introduced into equation (5.2) are, except for a sign, the Pfaffian of the matrices obtained from H cutting the ith and jth rows and also the ith and jth columns (Turnbull 1960). A general expression for the p_{ji} in this order with $i < j$ is

$$p_{ji} = h_{kl}h_{mn} - h_{km}h_{ln} + h_{kn}h_{lm} \tag{5.3}$$

where $(ijklmn)$ is an even permutation of (123456). Otherwise, we need to change the sign of the terms in the right-hand side. We are also considering in the above formula only $i < j$ for each h_{ij}.

We can form a matrix with the Pfaffians agregate p_{ij} that will be called hereafter the *Pfaffian* matrix associated with H which will be indicated by P. We define the Pfaffian matrix P such that it has the same symmetry as H, i.e. $(gP)^t = -gP$. The name Pfaffian for the matrix P is well justified since we have

$$PH = HP = \sqrt{\det H}\,I_6 \stackrel{\mathrm{def}}{=} \mathrm{Pfaff}\,H I_6. \tag{5.4}$$

The explicit expression for the Pfaffian of the matrix H (5.1) is given by

$$C_6 = (c\boldsymbol{e}\cdot\boldsymbol{b} - v_0\boldsymbol{a}\cdot\boldsymbol{b} + a_0\boldsymbol{b}\cdot\boldsymbol{v} + \boldsymbol{e}\cdot(\boldsymbol{a}\times\boldsymbol{v}))^2.$$

We are using a short-cut notation; $\boldsymbol{e} = (e_1, e_2, e_3)$ and $\boldsymbol{v}$, $\boldsymbol{b}$ and $\boldsymbol{a}$ are three-dimensional vectors from which we can compute the dot $\cdot$ and cross $\times$ product in the usual way.

Moreover, it is remarkable that the series for the odd powers can be rewritten by means of H, H^3 and P, avoiding the use of H^5. The advantage of working with the matrix P instead of H^5 is clear since P is second order in H. The above remark is based on the following equation that can be verified directly

$$H^5 = a_0H^3 + b_0H - \mathrm{i}\sqrt{c_0}P \tag{5.5}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. We remark that $c_0 = -\det H$, cf equation (4.1).

The above formula, as well as the formulae presented in the appendix, equations (A.4) and (A.5), were checked by a symbolic mathematical program.

The recurrence relation which comes from the Hamilton–Cayley theorem (4.1) can be obtained from the above equation. However, the converse is not true, i.e., we cannot obtain the latter equation from the Hamilton–Cayley theorem if $\det H = 0$, but the above equation holds even when $\det H = 0$.

The series for the odd powers given in equation (4.8) can be rewritten by means of H, H^3 and P. After substituting the above expression for H^5 and using equation (4.4) to replace a_0, b_0 and c_o by x, y and z, we find

$$\begin{aligned} w\sum\frac{H^{2i+1}}{(2i+1)!} &= [(z^4-x^4)y\sinh y + (y^4-z^4)x\sinh x + (x^4-y^4)z\sinh z]H \\ &\quad - \mathrm{i}[(x^2-z^2)xz\sinh y + (z^2-y^2)yz\sinh x + (y^2-x^2)xy\sinh z]P \\ &\quad + [(x^2-z^2)y\sinh y + (z^2-y^2)x\sinh x + (y^2-x^2)z\sinh z]H^3. \end{aligned} \tag{5.6}$$

Now, we shall discuss as a special case how the series for the Lorentz group $SO_+(1,3)$ presented in Zeni and Rodrigues (1990) can be obtained from the series for $SO_+(2,4)$. Let us write F for the matrix H (5.1) when a_μ, v_μ and c vanish. Therefore, the proper and orthochronous Lorentz transformations are generated by the matrix F. In this case, two of the eigenvalues $\{\pm z\}$ of F vanish since $\det F = 0$. Moreover, the product of the other two eigenvalues x and y can be written as $\boldsymbol{e}\cdot\boldsymbol{b} = \mathrm{i}, x, y$ since in this case $C_4 = (xy)^2 = -(\boldsymbol{e}\cdot\boldsymbol{b})^2$.

We also obtain a simpler recurrence relation for the powers of F; instead of the Hamilton–Cayley, equations (4.1) or (5.5) (Zeni and Rodrigues 1990), now we have

$$F^3 - (x^2+y^2)F - \mathrm{i}xyG = 0 \tag{5.7}$$

where G is the dual (Hodge) matrix obtained from F by changing $e_j \to b_j$ and $b_j \to -e_j$. The dual G has the following significant property

$$FG = GF = \mathrm{i}xyJ_4 = (\boldsymbol{e}\cdot\boldsymbol{b})J_4 \tag{5.8}$$

where $J_4 = \mathrm{diag}(1,1,1,1,0,0)$.

From the two equations above it is clear that we can express F^4 and F^3 by means of F^2, J_4 and F, G, respectively.

We remark that if we put $z = 0$ in equation (4.8), the coefficient of identity I_6 becomes equal to $w' = -x^2y^2(x^2 - y^2)$; the value of ω (4.5) when $z = 0$. Also, the coefficient of the Pfaffian matrix P, given by equation (5.6), vanishes in this case.

Substituting equation (5.7) for $F^3(= H^3)$ in equation (5.6) and for F^4 in equation (4.8), we can write the series for the 6×6 matrix F by means of J_4, F, F^2 and G as

$$\mathrm{e}^F = I_6 - J_4 + \left(\frac{x \sinh x - y \sinh y}{x^2 - y^2}\right) F + \left(\frac{y \sinh x - x \sinh y}{x^2 - y^2}\right) G + \left(\frac{\cosh x - \cosh y}{x^2 - y^2}\right) F^2 + \left(\frac{x^2 \cosh y - y^2 \cosh x}{x^2 - y^2}\right) J_4 \tag{5.9}$$

The series presented in Zeni and Rodrigues (1990, equation (6)) are obtained from the equation above by changing $y \to -\mathrm{i}y'$, except for the additive factor $I_6 - J_4$.

As a further special case, the generators of $SO(3)$ are given by the matrix F when $e_j = 0$, $j \in [1, 3]$. The exponential of a generator of $SO(3)$ (1.1) can be obtained from the formula above by putting $x = 0$, changing $y \to -\mathrm{i}\theta$, $J_4 \to J_3 = \mathrm{diag}(0, 1, 1, 1, 0, 0)$ and considering that $F = \theta \mathcal{L}_j n_j$.

6. Conclusions

In this article we presented a finite formula for the exponential of the Lie algebra to the conformal group $SO(2, 4)$ which is homomorphic to the special unitary group $SU(2, 2)$. This latter group has been used in spin-gauge transformations where the exponential can be used to determine the explicit form of the transformation of the Dirac matrices (Barut and McEwan 1984).

We plan to discuss the related result to the exponential map for the $SU(2, 2)$ group in a forthcoming paper and establish some connections with the works already existing in the literature for the unitary groups $SU(n)$, see, for example, Barnes and Delbourgo (1972) and Bincer (1990).

At this point, we recall that the use of other algebras, in particular Clifford algebras, can simplify the discussion; as seen in Zeni and Rodrigues (1992) where the Clifford algebra of spacetime was used to get the exponential map to $Spin_+(1, 3) \sim SL(2, C)$ in a very simple way, establishing a straightforward generalization of the first part of equation (1.1), this in turn is the exponential of a pure quaternion (Silva Leite (1993) obtained the exponential of octonions as another possible generalization of a quaternion exponential). We are particularly concerned with the Clifford algebra generated by the vector space $R^{2,4}$ since we have $Spin_+(2, 4) \sim SU(2, 2)$.

Our approach to obtaining the exponential can be used for every orthogonal group since equation (3.4) holds universally. From equation (3.4) it is possible to obtain an explicit formula for the exponential map from the Lie algebra in the connected component to the identity of the group, as we have achieved in this article for the $SO_+(2, 4)$ group. It is a remarkable result since the exponential map is usually presented only in the infinitesimal form and assumed to hold only in a neighbourhood of the identity (Barut and Razcka 1986, Miller Jr 1972).

Also from the finite formula for the group element we can easily discuss the group law and related subjects. In particular, it makes the Baker–Campbell–Hausdorff formula superfluous (Miller Jr 1972) since it provides an exact solution for problems related to this series (see Zeni and Rodrigues (1992) for a discussion related to the Lorentz group, $SL(2, C)$).

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Appendix. Eigenvalues with power series expansion

Here we present some expressions for the coefficients of the secular equation for general matrices and then we specialize our formulae for the generators of orthogonal groups. We remark that formulae for these coefficients are usually obtained by using the minors of the matrix (Turnbull 1960). Our approach is different and gives the coefficient by means of the trace of powers of the matrix.

We use the following formula for calculating the determinant (Miller Jr 1972):

$$\det(B-\lambda I_n) = (-\lambda)^n \exp\left[\operatorname{Tr}\ln\left(I_n - \frac{1}{\lambda}B\right)\right] = (-\lambda)^n \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!}\left[\sum_{l=1}^{\infty}\frac{1}{l\lambda^l}\operatorname{Tr} B^l\right]^m \tag{A.1}$$

where we make use of the series expansion for $\ln(1+x)$ and the exponential functions.

From this formula we can see directly that if the odd powers of the matrix B are traceless, e.g. the generators of orthogonal groups, we have only odd or even powers of the eigenvalue in the secular equation, according to whether n is odd or even (cf equation (2.10)).

Collecting terms of the same power in eigenvalue λ we get

$$\det(B-\lambda I_n) = (-\lambda)^n\left(1+\sum_{k=1}^{\infty}\frac{1}{\lambda^k}C_k\right). \tag{A.2}$$

Since the determinant of the secular equation has only positive powers in λ, the series in negative powers of λ is actually a finite series and we must have $C_k = 0$ for $k > n$. To gain an understanding of this, we recall the Hamilton–Cayley theorem which provides us with a relationship for the powers $k \geqslant n$ of a matrix.

The coefficients C_k can be written in the following way:

$$C_k = \frac{(-1)^k}{k!}(\operatorname{Tr} B)^k + \sum_{m=1}^{k-1}\sum_{p=1}^{m}\frac{(-1)^m}{m!}(\operatorname{Tr} B)^{m-p}\sum_{l_1=1}^{L_1}\cdots\sum_{l_{p-1}=1}^{L_{p-1}}\left(\prod_{i=1}^{p-1}\frac{1}{l_i}\operatorname{Tr} B^{l_i}\right)\frac{1}{l_p}\operatorname{Tr} B^{l_p} \tag{A.3}$$

with

$$L_j = k+j-m-1-\sum_{i=1}^{j-1} l_i \qquad \text{and} \qquad l_p = k+p-m-\sum_{i=1}^{p-1} l_i$$

The first three coefficients for a generic matrix $B_{n\times n}$ with $n \geqslant 3$ are

$$\begin{aligned} C_1 &= -\operatorname{Tr} B \\ C_2 &= \tfrac{1}{2}(\operatorname{Tr} B)^2 - \tfrac{1}{2}\operatorname{Tr} B^2 \\ C_3 &= -\tfrac{1}{3}\operatorname{Tr} B^3 + \tfrac{1}{2}\operatorname{Tr} B \operatorname{Tr} B^2 - \tfrac{1}{6}(\operatorname{Tr} B)^3. \end{aligned} \tag{A.4}$$

As mentioned before, if the odd powers of the matrix B are traceless, the coefficients C_k, for k odd, vanish. In this case, the formula for the coefficients (A.3) can be substantially simplified because the only non-vanishing terms are those with $m = p$.

We remark that the determinant of the matrix is given by the coefficient C_n. For example, for the generators of the $O(p,q)$ groups with $p+q=6$, we have that

$$C_6 = -\tfrac{1}{48}(8\operatorname{Tr} B^6 - 6\operatorname{Tr} B^2 \operatorname{Tr} B^4 + (\operatorname{Tr} B^2)^3). \tag{A.5}$$

Finally, we remark that if we represent a second-rank tensor by a matrix, the 'principal' invariants of the tensor are just the coefficients of the secular equation of the matrix (Landau and Lifshitz 1951). The most common invariants related to a matrix are the trace, the first coefficient of the secular equation and the determinant, i.e. the last coefficient in the secular equation. However, all the coefficients present in the secular equation are also invariant. This becomes clear from the formula above since these coefficients are expressed by means of the trace.

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Zeni J R and Rodrigues W A 1990 *Hadronic J.* **13** 317. Equation (6d) is misprinted, it should read $g_2(x, y) = (y \sinh x - x \sinh y)/|z|^2$.

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